

Set systems without a 3-simplex

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Abstract

A 3-simplex is a collection of four sets A_1, \dots, A_4 with empty intersection such that any three of them have nonempty intersection. We show that the maximum size of a set system on n elements without a 3-simplex is $2^{n-1} + \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2}$ for all $n \geq 1$, with equality only achieved by the family of sets either containing a given element or of size at most 2. This extends a result of Keevash and Mubayi, who showed the conclusion for n sufficiently large.

1 Introduction

Throughout this paper X will be an n -element set. For an integer $i \geq 0$, let $X^{(i)} = \{A \subseteq X : |A| = i\}$, and $X^{(\leq i)} = \cup_{0 \leq j \leq i} X^{(j)}$. If $\mathcal{F} \subseteq X^{(\leq n)}$ and $x \in X$, we let $\mathcal{F}_x = \{A \in \mathcal{F} : x \in A\}$ and $\mathcal{F} - x = \mathcal{F} \setminus \mathcal{F}_x$.

A d -dimensional simplex, or d -simplex, is a collection of $d+1$ sets A_1, \dots, A_{d+1} such that $\cap_{i=1}^{d+1} A_i = \emptyset$ but $\cap_{i \neq j} A_i \neq \emptyset$ for $1 \leq j \leq d+1$. For positive integers n, d, r , let

$$\begin{aligned} f(n, d) &= \max\{\mathcal{F} \subseteq X^{(\leq n)} : \mathcal{F} \text{ is } d\text{-simplex-free}\}, \text{ and} \\ f_r(n, d) &= \max\{\mathcal{F} \subseteq X^{(r)} : \mathcal{F} \text{ is } d\text{-simplex-free}\}. \end{aligned}$$

The problem of determining $f(n, d)$ and $f_r(n, d)$ can be traced to some of the most fundamental results in extremal combinatorics. As a 1-simplex is a pair of nonempty disjoint sets, it is easy to see that $f(n, 1) = 2^{n-1} + 1$, while the solution to determining $f_r(n, 1)$ comes from the celebrated Erdős-Ko-Rado Theorem:

Theorem 1 (Erdős-Ko-Rado [2]). *Let $n \geq 2r$ and suppose $\mathcal{F} \subseteq X^{(r)}$ is intersecting: then $|\mathcal{F}| \leq \binom{n-1}{r-1}$. If $n > 2r$ and equality holds, then $\mathcal{F} = X_x^{(r)}$ for some $x \in X$.*

For $r = d = 2$, the forbidden family is a triangle (in graphs), and thus $f_2(n, 2) = \lfloor n^2/4 \rfloor$, a special case of Turán's theorem and a cornerstone of extremal graph theory. Erdős later posed the question of determining the size of the largest r -uniform hypergraph without a triangle (2-simplex), i.e. $f_r(n, 2)$. Chvátal [1] solved the case $r = 3$ by showing the stronger result that for $n \geq r + 2 \geq 5$, $f_r(n, r-1) = \binom{n-1}{r-1}$ with equality only for $\mathcal{F} = X_x^{(r)}$ for some $x \in X$. Chvátal further conjectured the following:

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Conjecture 1 (Chvátal [1]). *Let $r \geq d + 1 \geq 3$, $n \geq r(d + 1)/d$, and $\mathcal{F} \subseteq X^{(r)}$ with no d -simplex. Then $|\mathcal{F}| \leq \binom{n-1}{r-1}$, with equality only if $\mathcal{F} = X_x^{(r)}$ for some $x \in X$.*

Progress was made on the $d = 2$ case over a number of years before it was finally settled by Mubayi and Verstraëte (see [7] for the result and further references). For $d \geq 3$, Frankl and Füredi [4] established Conjecture 1 for n sufficiently large (see also [5]), and, more recently, Keevash and Mubayi [6] confirmed it when r and $n/2 - r$ are bounded away from 0.

Erdős also posed the question of forbidding triangles in nonuniform systems, which was answered by Milner (unpublished), who showed $f(n, 2) = 2^{n-1} + n$ for all $n \geq 1$. Short proofs of the bound were also found by Lossers [3] and by Mubayi and Verstraëte [7] - the latter result establishing that the unique extremal family consists of all sets either containing a given element or of size at most 1.

For $d \geq 3$, Keevash and Mubayi completely determined $f(n, d)$ and the extremal family for n sufficiently large:

Theorem 2 (Keevash and Mubayi [6]). *Let $d \geq 2$, and suppose $\mathcal{F} \subseteq X^{(\leq n)}$ is d -simplex-free, where n is sufficiently large. Then $|\mathcal{F}| \leq 2^{n-1} + \sum_{i=0}^{d-1} \binom{n-1}{i}$, with equality if and only if $\mathcal{F} = X_x^{(\leq n)} \cup (X \setminus \{x\})^{(\leq d-1)}$ for some $x \in X$.*

The proof of Theorem 2 for $d \geq 3$ relies on a stability result that in turn relies on their solution to the uniform problem mentioned above. Our contribution is to completely determine $f(n, 3)$ and the associated extremal family using a simpler inductive argument.

Theorem 3. *For $n \geq 1$, suppose $\mathcal{F} \subseteq X^{(\leq n)}$ is 3-simplex-free. Then $|\mathcal{F}| \leq 2^{n-1} + \sum_{i=0}^2 \binom{n-1}{i}$, with equality if and only if $\mathcal{F} = X_x^{(\leq n)} \cup (X \setminus \{x\})^{(\leq 2)}$ for some $x \in X$.*

2 The Proof of Theorem 3

For $d \geq 1$ and $n, k \geq 0$, let $f(n, d, k)$ be the maximum size of a d -simplex-free family $\mathcal{F} \subseteq X^{(\leq n-k)}$, and let $g(n, d, k)$ be the maximum size of such a family \mathcal{F} with $\mathcal{F} \cap X^{(n-k)} \neq \emptyset$. Thus $f(n, d, k) = \max_{j \geq k} g(n, d, j)$, and as X cannot lie in a d -simplex in \mathcal{F} , $f(n, d) = 1 + f(n, d, 1)$. We begin our arguments with a simple lemma.

Lemma 1. *Let $n \geq k \geq 1$ and $d \geq 2$. Then*

$$g(n, d, k) \leq f(n - k, d) + \sum_{i=1}^k \binom{k}{i} f(n - k, d - 1, i). \quad (1)$$

Proof. Let $\mathcal{F} \subseteq X^{(\leq n-k)}$ be d -simplex-free with $|\mathcal{F}| = g(n, d, k)$ and $\max_{A \in \mathcal{F}} |A| = n - k$, and fix a $Y \in \mathcal{F}$ with $|Y| = n - k$. Let $Z = X \setminus Y$, and for every $W \subseteq Z$, let $\mathcal{F}_W = \{A \cap Y : A \in \mathcal{F}, A \cap Z = W\}$, so $|\mathcal{F}| = \sum_{W \subseteq Z} |\mathcal{F}_W|$.

Clearly \mathcal{F}_\emptyset must be d -simplex-free, so $|\mathcal{F}_\emptyset| \leq f(n - k, d)$. Now, fix any nonempty $W \subseteq Z$. If \mathcal{F}_W contains a $(d - 1)$ -simplex A_1, \dots, A_d , then letting $B_i = A_i \cup W$ for $1 \leq i \leq d$ and $B_{d+1} = Y$, the B_i form a d -simplex in \mathcal{F} , a contradiction. By the choice of Y , it follows that

every $A \in \mathcal{F}_W$ has size at most $n - k - |W|$, and hence $|\mathcal{F}_W| \leq f(n - k, d - 1, |W|)$ and the result follows. \square

We next show the following simple result for the $d = 1$ case, whose proof we include for completeness.

Claim 1. *For every $n \geq k \geq 1$, $g(n, 1, k) = f(n, 1, k) = 2^{n-1} - \sum_{j=1}^{k-1} \binom{n-1}{j}$.*

Proof. Let $\mathcal{F} \subseteq X^{(\leq n-k)}$ be 1-simplex-free with $\max_{A \in \mathcal{F}} |A| = n - k$ and $|\mathcal{F}| = g(n, 1, k)$. Let \mathcal{P} be a partition of $\{1, 2, \dots, n - k\}$ into singletons $\{i\}$ with $i \leq \frac{n}{2}$ and pairs $\{i, n - i\}$ with $i < n - i$. Finally, let $\mathcal{F}^i = \mathcal{F} \cap X^{(i)}$.

For every singleton $\{i\} \in \mathcal{P}$, by Erdős-Ko-Rado, $|\mathcal{F}^i| \leq \binom{n-1}{i-1} = \binom{n-1}{n-i}$. For every pair $\{i, n-i\} \in \mathcal{P}$, as $\mathcal{F} \setminus \{\emptyset\}$ is intersecting, $|\mathcal{F}^i| + |\mathcal{F}^{n-i}| \leq \binom{n}{i} = \binom{n-1}{i-1} + \binom{n-1}{i} = \binom{n-1}{n-i} + \binom{n-1}{n-(n-i)}$. As $|\mathcal{F}^0| \leq 1$, it follows that

$$|\mathcal{F}| \leq \sum_{i=0}^{n-k} |\mathcal{F}^i| \leq 1 + \sum_{i=1}^{n-k} \binom{n-1}{n-i} = \binom{n-1}{0} + \sum_{i=k}^{n-1} \binom{n-1}{i} = 2^{n-1} - \sum_{i=1}^{k-1} \binom{n-1}{i}.$$

To see that equality holds in the bound, let $\mathcal{F} = X_x^{(\leq n-k)} \cup \{\emptyset\}$ for any $x \in X$. We also note that for $k \geq 2$, equality implies $k = n$ or $|\mathcal{F}^1| = 1$ and hence this is the unique extremal family. \square

Next, we prove a slight strengthening of Milner's result on triangle-free set systems.

Lemma 2. *For all $n \geq 1$,*

$$f(n, 2, 1) = 2^{n-1} + \binom{n-1}{1}, \quad (2)$$

$$f(n, 2, 2) \leq 2^{n-1} + 1, \text{ and} \quad (3)$$

$$f(n, 2, 3) \leq 2^{n-1}. \quad (4)$$

In particular, $f(n, 2) = 2^{n-1} + \binom{n-1}{0} + \binom{n-1}{1}$. Moreover, if $\mathcal{F} \subseteq X^{(\leq n)}$ is triangle-free and $|\mathcal{F}| = f(n, 2)$, then $\mathcal{F} = X_x^{(\leq n)} \cup (X \setminus \{x\})^{(\leq 1)}$ for some $x \in X$.

Proof. Our proof is by induction on n ; it is easy to verify for $n \leq 3$, so suppose $n \geq 4$. As $g(n, 2, n) = 1$, let $1 \leq k \leq n - 1$. Applying Lemma 1 and Claim 1,

$$\begin{aligned} g(n, 2, k) &\leq f(n - k, 2) + \sum_{i=1}^k \binom{k}{i} f(n - k, 1, i) \\ &\leq f(n - k, 2) + \sum_{i=1}^k \binom{k}{i} \left(2^{n-k-1} - \sum_{j=1}^{i-1} \binom{n-k-1}{j} \right) \\ &\leq f(n - k, 2) + (2^k - 1) 2^{n-k-1} - \binom{k}{2} \binom{n-k-1}{1} \\ &= 2^{n-1} + \binom{n-k-1}{0} + \binom{n-k-1}{1} \left(1 - \binom{k}{2} \right). \end{aligned} \quad (5)$$

From (5), it follows that $g(n, 2, k) \leq 2^{n-1} + 1$ for $2 \leq k \leq n-1$, with equality only possible at $k = 2$ or $k = n-1$: as $n \geq 4$, $n-1 > n/2$, so $g(n, 2, n-1) \leq 2^{n-1}$ and (3) and (4) follow. The upper bound in (2) also follows from (5), and the lower bound from the conjectured extremal family.

Suppose now that $\mathcal{F} \subseteq X^{(\leq n)}$ is a triangle-free family with $|\mathcal{F}| = 2^{n-1} + \binom{n-1}{0} + \binom{n-1}{1} > 1 + f(n, 2, 2)$: it follows that there is a $Y \in \mathcal{F}$ with $|Y| = n-1$. Let z be the unique element in $X \setminus Y$. Define $\mathcal{F}^1 = \mathcal{F} - z$ and $\mathcal{F}^2 = \{A \cap Y : A \in \mathcal{F}_z\}$. As equality holds in (2) (with $k = 1$), it follows from the proof of Lemma 1 that \mathcal{F}^1 is triangle-free, \mathcal{F}^2 is 1-simplex-free, $|\mathcal{F}^1| = f(n-1, 2)$ and $|\mathcal{F}^2| = 1 + f(n-1, 1, 1) = f(n-1, 1)$. By the induction hypothesis, $\mathcal{F}^1 = Y_y^{(\leq n-1)} \cup (Y \setminus \{y\})^{(\leq 1)}$ for some $y \in Y$.

Suppose there is an $A \in \mathcal{F}$ with $|A| \geq 2$ and $y \notin A$: then $z \in A$, so $A = \{z, w_1, w_2, \dots, w_s\}$ for some $s \geq 1$. If $s \geq 2$, then the sets $\{y, w_1\}$, $\{y, w_2\}$ and A form a triangle in \mathcal{F} , a contradiction. If $s = 1$, then $\{w_1\} \in \mathcal{F}^2$, implying that $\mathcal{F}^2 = Y_{w_1}^{(\leq n-1)} \cup \{\emptyset\}$. As $|Y| \geq 3$, let $w_2 \in Y \setminus \{w_1, y\}$: then $\{z, w_1, y\}$, $\{z, w_1, w_2\}$, $\{w_2, y\}$ lie in \mathcal{F} and form a triangle, a contradiction. Therefore $\mathcal{F} \subseteq X_y^{(\leq n)} \cup (X \setminus \{y\})^{(\leq 1)}$ and hence equality holds. \square

Now that the pieces are in place, we prove the main result.

Proof of Theorem 3. We use the same inductive approach as in the proof of Lemma 2. We note that the result is trivial for $n < 4$, and for $n = 4$ the only restriction is that a single 3-element set must be missing. Therefore, assume $n \geq 5$: let $\mathcal{F} \subseteq X^{(\leq n)}$ be 3-simplex-free with $|\mathcal{F}| \geq 2^{n-1} + \binom{n-1}{0} + \binom{n-1}{1} + \binom{n-1}{2}$. Let $Y \in \mathcal{F} \setminus \{X\}$ have maximum size, and let $k = n - |Y|$. Then by Lemmas 1 and 2,

$$\begin{aligned}
|\mathcal{F}| - 1 &\leq g(n, 3, k) \\
&\leq f(n-k, 3) + \sum_{i=1}^k \binom{k}{i} f(n-k, 2, i) \\
&\leq f(n-k, 3) + (2^k - 1)2^{n-k-1} + \binom{k}{1} \binom{n-k-1}{1} + \binom{k}{2} \\
&= f(n-k, 3) + (2^k - 1)2^{n-k-1} + \binom{n-1}{2} - \binom{n-k-1}{2} \\
&= 2^{n-1} + \binom{n-k-1}{0} + \binom{n-k-1}{1} + \binom{n-1}{2} \\
&= 2^{n-1} + \binom{n-k}{1} + \binom{n-1}{2},
\end{aligned}$$

which by our lower bound on $|\mathcal{F}|$ implies equality holds throughout and $k = 1$.

Let z be the unique element in $X \setminus Y$ and let $\mathcal{F}^1 = \mathcal{F} - z$ and $\mathcal{F}^2 = \{A \setminus \{z\} : A \in \mathcal{F}_z\}$: then \mathcal{F}^1 is 3-simplex-free and of size $f(n-1, 3)$, and \mathcal{F}^2 is triangle-free and of size $f(n-1, 2, 1) + 1 = f(n-1, 2)$. By Lemma 2 and the induction hypothesis, there exist $y_1, y_2 \in Y$ such that $\mathcal{F}^1 = Y_{y_1}^{(\leq n-1)} \cup (Y \setminus \{y_1\})^{(\leq 2)}$ and $\mathcal{F}^2 = Y_{y_2}^{(\leq n-1)} \cup (Y \setminus \{y_2\})^{(\leq 1)}$. As every set in \mathcal{F}^2 of size at most 1 corresponds to a set of size at most 2 in \mathcal{F} , it suffices to show that $y_1 = y_2$, so suppose otherwise. As $n \geq 5$, $|Y| \geq 4$, so let $w_1, w_2 \in Y \setminus \{y_1, y_2\}$:

then the sets $\{y_1, y_2, w_1\}, \{y_1, y_2, w_2\}, \{y_1, w_1, w_2\}$ and $\{z, y_2, w_1, w_2\}$ all lie in \mathcal{F} and form a 3-simplex, a contradiction. \square

3 Concluding Remarks

Complications arise in attempting to extend this method to forbidding d -simplices with $d \geq 4$, the chief among them following from the fact that for $k \geq 2$, (1) is not, in general, sharp. To see this and illustrate the difficulty with the $d = 4$ case, note that by the extremal family, $f(n, 3, 2) \geq 2^{n-1} + \binom{n-1}{2}$ for $n \geq 4$. With similar calculations as above, this implies the best upper bound on $g(n, 4, 2)$ guaranteed by (1) is $2^{n-1} + \binom{n-1}{2} + \binom{n-1}{3} + \binom{n-3}{2}$, which is greater than $2^{n-1} + \binom{n-1}{1} + \binom{n-1}{2} + \binom{n-1}{3}$ for all $n \geq 8$.

However, we suspect that, in general and for most k , the family $X_x^{(\leq n-k)} \cup (X \setminus \{x\})^{(\leq d-1)}$ determines $f(n, d, k)$ and $g(n, d, k)$:

Conjecture 2. *For $d \geq 2$ and $1 \leq k \leq n - d - 1$, if $\mathcal{F} \subseteq X^{(\leq n-k)}$ is d -simplex-free, then*

$$|\mathcal{F}| \leq 2^{n-1} + \sum_{i=0}^{d-1} \binom{n-1}{i} - \sum_{i=0}^{k-1} \binom{n-1}{i},$$

with equality if and only if $\mathcal{F} = X_x^{(\leq n-k)} \cup (X \setminus \{x\})^{(\leq d-1)}$ for some $x \in X$.

We mention that the proof of Theorem 2 yields that Conjecture 2 holds for $k \leq d$ provided n is sufficiently large.

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